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THE SOLUTION OF NONLINEAR TIME-DEPENDENT PROBLEMS USING MODAL COORDINATES

ABSTRACT An algorithm for the solution of a class of nonlinear time-dependent problems using reduced modal coordinates is proposed and implemented. Example problems in one space dimension are modelled, and comparisons made to exact solutions. The results of these examples are used to make suggestions for further research.
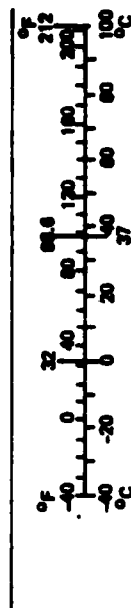
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METRIC CONVERSION FACTORS

Approximate Conversions to Metric Measures

Symbol	When You Know	Multiply by	To Find	Symbol
		LENGTH		
in	inches	*2.5	centimeters	cm
ft	feet	30	centimeters	cm
yd	yards	0.9	meters	m
mi	miles	1.6	kilometers	km
		AREA		
in²	square inches	6.5	square centimeters	cm²
ft²	square feet	0.09	square meters	m²
yd²	square yards	0.8	square meters	m²
mi²	square miles	2.6	square kilometers	km²
	acres	0.4	hectares	ha
		MASS (weight)		
oz	ounces	28	grams	g
lb	pounds	0.45	kilograms	kg
	short tons (2,000 lb)	0.9	tonnes	t
		VOLUME		
tsp	teaspoons	5	milliliters	ml
Tbsp	tablespoons	15	milliliters	ml
fl oz	fluid ounces	30	milliliters	ml
c	cups	0.24	liters	l
pt	pints	0.47	liters	l
qt	quarts	0.95	liters	l
gal	gallons	3.8	liters	l
ft³	cubic feet	0.03	cubic meters	m³
yd³	cubic yards	0.76	cubic meters	m³
		TEMPERATURE (exact)		
°F	Fahrenheit temperature	5/9 (after subtracting 32)	Celsius temperature	°C

*1 in = 2.54 (exactly). For other exact conversions and more detailed tables, see NBS Misc. Publ. 286, Units of Weights and Measures, Price $2.25, SD Catalog No. C13.10:286.

Approximate Conversions from Metric Measures

Symbol	When You Know	Multiply by	To Find	Symbol
		LENGTH		
mm	millimeters	0.04	inches	in
cm	centimeters	0.4	inches	in
m	meters	3.3	feet	ft
m	meters	1.1	yards	yd
km	kilometers	0.6	miles	mi
		AREA		
cm²	square centimeters	0.16	square inches	in²
m²	square meters	1.2	square yards	yd²
km²	square kilometers	0.4	square miles	mi²
ha	hectares (10,000 m²)	2.5	acres	
		MASS (weight)		
g	grams	0.035	ounces	oz
kg	kilograms	2.2	pounds	lb
t	tonnes (1,000 kg)	1.1	short tons	
		VOLUME		
ml	milliliters	0.03	fluid ounces	fl oz
l	liters	2.1	pints	pt
l	liters	1.06	quarts	qt
l	liters	0.26	gallons	gal
m³	cubic meters	35	cubic feet	ft³
m³	cubic meters	1.3	cubic yards	yd³
		TEMPERATURE (exact)		
°C	Celsius temperature	9/5 (then add 32)	Fahrenheit temperature	°F

°F -40   0   32   40   80   98.6   120   160   200   212
°C -40   -20   0   20   37   40   60   80   100

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An algorithm for the solution of a class of nonlinear time-dependent problems using reduced modal coordinates is proposed and implemented. Example problems in one space dimension are modeled, and comparisons made to exact solutions. The results of these examples are used to make suggestions for further research.

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I. Introduction

The Modal Superposition Method is a useful algorithm for the solution of linear problems in structural dynamics. The basic idea of the method is to transform the coupled equations of motion into an uncoupled form. This change of basis is equivalent to diagonalization of the underlying matrix differential equation, and has the effect of trading one complicated n-dimensional system of ODE for n simple (scalar) ODE. Typically, this change of basis to modal coordinates is very efficient computationally, and linear problems are generally solved in this manner. Often, not all the modes participate in the solution, and so the problem may be projected onto a smaller subspace, with a corresponding decrease in computational cost.

The principle of superposition does not apply to nonlinear problems, so Modal Superposition is not strictly valid in a nonlinear setting. The method can be extended to this more difficult case by considering the linearization of the nonlinear problem in a local region. This linearized problem can be used to generate modal coordinates, and these modes can be used as generalized coordinates in the solution of the nonlinear problem. The presence of nonlinearities has the effect of introducing coupling among the modal equations of motion but the reduction in size of the problem obtained by projection to a smaller modal basis typically justifies the increased cost due to this coupling. In addition, the knowledge that off-diagonal terms will exist in the modal problem means that the exact modes (the ones that diagonalize the linearized problem) need not be found: an approximation will do, since the effect of the error is to add

1

off-diagonal terms to the (already coupled) modal problem.

The scope of this report is to propose a general algorithm for the solution of a class of nonlinear problems in structural dynamics using a modal representation. The development of some of the underlying linear theory will be presented, followed by generalizations to nonlinear problems. These topics will be used to develop an algorithm for the nonlinear case, and several one-dimensional examples will be presented. This research is a starting point in an attempt to produce an adaptive algorithm for the solution of nonlinear problems, and so a number of topics for further development will be presented.

It is assumed that the reader has a basic knowledge of structural dynamics, finite element methods, and matrix spectral theory. Sufficient detail on these topics can be found in references [1], [2], and [3].

Some consideration of semantics should be noted: in the following development, a "mode" is taken to be a generalized coordinate vector that reflects some aspect of the vibrational behavior of the structure, but not necessarily an exact eigenvector of the mathematical model. (This is an ambiguous definition, but the word "mode" is considerably more imprecise than "eigenvector", with which it is often taken to be synonymous.) Different modes are taken to satisfy some type of orthogonality requirement (usually with respect to the finite element mass matrix), and to have unit length in some appropriate norm, but they do not have to produce a diagonal form of the equations of motions, as in linear theory. This convention is merely for the convenience of lumping classes of generalized

coordinates into the term "mode", instead of referring to "exact eigenvectors, Ritz vectors, Lanczos approximations to eigenvectors", etc.

Finally, all important symbols are tabulated in Appendix 1.

II. Linear Modal Analysis

The equations of motion for a linear-elastic structure are a set of partial differential equations involving both spatial and temporal derivatives. These PDE can be cast in a semi-analytic form by performing a finite element discretization in the space domain. This process turns the spatial differential terms into an algebraic set of equations. The time derivatives are left in continuous form, and the resulting system of ordinary differential equations can be written for an undamped system as:

$$Kx + M\ddot{x} = f \qquad\qquad Eqn\ 1$$

where K and M are the finite element stiffness and mass matrices, x is the vector of nodal displacements, and f is a load vector which is a function of time. The modal coordinates of this system can be found by solving the generalized eigenvalue problem:

$$(K - w^2M)u = 0 \qquad\qquad Eqn\ 2$$

The pair (K,M) is termed a matrix pencil (this term arises from the study of geometric optics). In general, K and M are positive-definite matrices, and so the eigenvalues w^2 are real and positive. Under these conditions, a complete set of eigenvectors exists that are orthonormal with respect to the mass matrix:

$$U = [u_1, u_2, \ldots u_n] \qquad \text{Eqn 3}$$

$$U^T M U = I \qquad U^T K U = D = \text{diag}(w_i^2)$$

The change of basis to modal coordinates z is given by

$$Uz = x \qquad \text{Eqn 4}$$

In this coordinate system, the ODE are uncoupled, so that Eqn 1 becomes:

$$\ddot{z}_i + w_i^2 z_i = u_i^T f \qquad \text{Eqn 5}$$

These equations can easily be integrated to give the modal coordinates z, which are then transformed using Eqn 4 to give the desired solution. Any numerical integration scheme appropriate for a single-degree-of-freedom system, such as the Duhamel Integral (see [1], pgs 100-108) can be used to solve the modal equations.

In practice, the solution of linear problems generally includes a damping term $C\dot{x}$ to account for energy loss in the structure. In order to diagonalize K, C, and M simultaneously, the damping matrix C is generally taken to be some combination of powers of K and M. (In many nonlinear problems, which are the ultimate scope of this research, the stiffness $K(x,t)$ typically includes nonconservative terms, and so the construction of a damping matrix is not considered further.) The presence of damping in a linear system causes the transient free-vibration response of the system (corresponding to the homogenous solution of Eqn 1) to decay, leaving the steady-state forced response induced by the load f (the nonhomogenous solution of Eqn 1). In the following discussion, the "response" of the structure is taken to be the steady-state response.

The loading term $u_i^T f$ represents a generalized load acting on the ith modal coordinate. If u_i and f are nearly orthogonal, then this load is small relative to other generalized modal loads. In this case, the ith mode does not appreciably participate in the solution, and can be neglected in the analysis. Typically, in the modelling of a complex structure, most modes do not participate, and thus a reduced problem can be solved by projecting the solution onto a p-dimensional subspace spanned by the p "most important" modes:

$$x = U_p z \qquad\qquad\qquad \text{Eqn 6}$$

In this case, the columns of U_p (an nxp matrix) are the p dominant orthonormal modes, and z (a px1 vector) is the reduced set of modal coordinates. It is important to observe that $U_p^T U_p = I_p$, the identity

matrix for p-dimensional space, but that $U_p U_p^T \neq I_n$, unless $p = n$. In spite of this last fact, U_p is still often termed an orthogonal matrix, and this terminology will be used herein.

The load vector f can be decomposed into orthogonal components $f = g + h$ where g lies in the column space (or range) of U_p, and $g^T h = 0$. The vectors g and h are defined by projecting f onto the range of U_p:

$$g = U_p U_p^T f \qquad h = (I - U_p U_p^T) f \qquad \qquad \text{Eqn 7}$$

The Pythagorean theorem implies that $g^T g + h^T h = f^T f$ (i.e. $\|g\|^2 + \|h\|^2 = \|f\|^2$), so that it is to be hoped that $g^T g \gg h^T h$ in order that the reduced set of coordinates gives a good approximation to the load. Finally, in preparation for subsequent development, it should be noted that the projection defined by Eqns 6 and 7 is valid for any matrix U_p with orthonormal columns, but that the particular U_p formed using the eigenvectors of the pencil (K, M) diagonalizes the linear problem.

Once the important modes of the linear problem have been found, the scalar ODE of Eqn 5 can be integrated to give the solution. A problem occurs when the time scales defined by the modal frequencies w_i span several orders of magnitude:

$$\max_i(w_i) \gg \min_i(w_i)$$

In this case, any step size appropriate for numerical integration of a

"fast" problem ($\max_i(w_i)$) will be too small for efficient integration of a "slow" problem ($\min_i(w_i)$), and a step size appropriate for a slow problem is inaccurate for a fast one. If the load function f defines another time scale (i.e. the duration or the rapid variation of an earthquake record), then the situation is even more complicated. In any case, a wide range of time scales exists in the problem, and any computational scheme must accomodate this difficulty. Differential equations characterized by this range of scales are referred to as stiff: in this case, the stiffness is inherent in the spread of the eigenvalues w_i^2 of the pencil (K,M). In a linear modal analysis, the stiff nature of the equations requires selection of an appropriate step size for integration of the modal equations: since these equations are uncoupled, these step sizes may be chosen independently. An algorithm that does not completely decouple the modes may require short steps for a slow mode if it remains coupled to a faster one. Thus, there is strong incentive to diagonalize (or nearly-diagonalize) any linear problem, except in very special circumstances (such as an extremely short-duration loading and response, where the cost of finding the modes cannot be amortized over sufficient time steps to remain competitive).

Two algorithms for generation and solution of reduced equations for linear problems have appeared recently in the literature. Although these algorithms are motivated from different basic principles, in fact they are closely related. Each represents an efficient alternative to solution of the problem using exact eigenvectors.

The first approach is due to Wilson, et. al. [4][5], who suggests using so-called "Ritz vectors" to achieve a reduced set of coordinates. The identification of these Ritz vectors proceeds from a physical argument involving the solution of a static problem, followed by corrections accounting for dynamic terms missed in previous steps. (The reader is referred to [4] for the details. The following summary involves some notational changes from this original reference.) In summary, the Ritz vectors q_i are obtained using the recurrence:

1) $Kx_1 = f$ (f = equivalent static load)

2) $q_1 = x_1/(x_1^T M x_1)$

3) For $i = 2$ to p

3a) $Kx_i = Mx_{i-1}$

3b) $y_i = x_i - \sum_{j<i}(q_j^T M x_i) q_j$

3c $q_i = y_i/(y_i^T M y_i)$

The nxp matrix $Q_p = [q_1|q_2| \ldots |q_p]$ is (in the absence of round-off error) an orthogonal matrix. In practice, step 3b of the algorithm is badly affected by round-off error in a low-precision computational environment, and needs to be replaced with a more robust orthogonalization scheme (such as the one found in [3], pg 105-110), in order to achieve orthogonality of the columns of Q_p to working precision. If enough vectors are included in Q_p, these reduced coordinates give excellent results: in the example problems studied in [4], this algorithm was clearly superior to the use of exact eigenvectors. The authors suggest that further savings could be

realized by diagonalizing the reduced stiffness $Q_p^T K Q_p$ before numerical integration is performed. Since $p \ll n$, this last step is very inexpensive.

Examination of this algorithm reveals that the Ritz vectors are chosen to be an orthonormal basis for the space spanned by the set $\{K^{-1}f, (K^{-1}M)K^{-1}f, (K^{-1}M)^2 K^{-1}f, \ldots , (K^{-1}M)^{p-1}K^{-1}f\}$. (This subspace is termed the Krylov subspace of order p, generated by the matrix $K^{-1}M$ and the vector $K^{-1}f$: reference [3] contains several excellent chapters on the utility of these Krylov subspaces.) The algorithm of references [4] and [5] therefore generate these Ritz vectors by direct calculation of an orthonormal basis for this subspace, with full orthogonalization (step 3b) at each step.

The second approach for generation of a reduced problem is due to Nour-Omid and Clough [6]. This method uses the Lanczos Algorithm (refer to [3] or to [7] for a detailed derivation) to generate Lanczos vectors: these vectors define the reduced coordinate set for the problem. The Lanczos Algorithm involves the use of a three-term recurrence to generate an orthonormal basis for the Krylov subspace mentioned in the last paragraph. This recurrence (even after modifications are made to reduce the effect of round-off error) is considerably less expensive than the orthogonalization scheme used in [4][5], and results in a tridiagonal form for the equations of motion. The resulting reduced problem can easily be diagonalized, or a tridiagonal formulation for the numerical integration of the equations of motion can be used. In either case, the computational cost is much less than any method for solving the full (unreduced) problem.

III. Nonlinear Modal Analysis

A common class of nonlinear problems in structural dynamics occurs when the stiffness and/or mass matrices are functions of the displacement x (or its temporal derivatives). In this case, the equilibrium equations given by Eqn 1 become:

$$K(x,\dot{x},\ddot{x},t)x + M(x,\dot{x},\ddot{x},t)\ddot{x} = f(x,\dot{x},\ddot{x},t) \qquad \text{Eqn 8}$$

The most typical example of this class of problems is considerably simpler:

$$K(x,t)x + M\ddot{x} = f(x,t) \qquad \text{Eqn 9}$$

In this case, the mass matrix is constant, and the stiffness and force depend only on x and t. The following development assumes that the nonlinearities can be cast in the form of Eqn 9: the more general case given by Eqn 8 can be analyzed in a similar manner.

For a given configuration $x_0 = x(t_0)$, the pencil $(K(x_0,t_0)\ M)$ can be used to generate a set of modes. These modes could be the exact eigenvectors that diagonalize this pencil or, as in the last section, could be Lanczos vectors that tridiagonalize $MK^{-1}M$. Since the problem is nonlinear, it is not expected that this diagonal (or tridiagonal) structure will be preserved for $x \neq x_0$. (To be more precise, the nonlinear problem can be linearized at x_0, and a set of reduced modal coordinates extracted.)

The primary questions in such a local reduction are the number of reduced coordinates required to gain an acceptable approximation in a region near x_0, and the size of the region over which the approximation is valid. Assuming that a representative load vector (or some generalization of a participation factor) can be found, the methodology of references [4], [5], and [6] can be used to determine the number of coordinates required at x_0. The magnitudes of the projected load and the error involved in this projection can be monitored to insure that enough modes have been included to maintain accuracy. Some measure of off-diagonal strength in the reduced equations, such as

$$\sum_{j<i} (r_{ij})/(r_{ii} r_{jj})^{\frac{1}{2}}$$

(where r_{ij} is an element of the reduced stiffness) can be used to determine whether the modal coupling is too strong (as in solving stiff problems). All these topics are currently being studied, and the results will be presented in future publications.

For some problems (see reference [8] and the examples presented in the following sections), good results have been obtained by taking x_0 and t_0 to be zero (i.e., using the initial configuration to generate mode shapes) and applying these same reduced coordinates throughout the entire nonlinear problem. On the other hand, when the nonlinearities involve large inelastic effects, it is expected that the modes of vibration will need to be updated at some stages of the solution process.

IV. Proposed Algorithm for Reduced Coordinate Analysis

Using the direct integration of the equations of motion as a model, and including some scheme (such as those just presented) to generate a set of reduced coordinates, the following skeletal algorithm is presented as an outline of a general solution scheme for nonlinear problems of the form given in Eqn 9:

1) Initialize:

1a)   set initial conditions on x, $\dot{x}$, t, f

1b)   calculate initial values of K, $\ddot{x}$, M

1c)   evaluate mode shapes Q_p

2) For $t = t_0$ to $t = t\_max$

2a)   update modes, if necessary

2b)   form reduced problem $Rz + \ddot{z} = g$

2c)   integrate reduced problem to end of time step

2d)   if equilibrium satisfied then done with step

         else apply iteration strategy and repeat from (2b) or (2a)

This algorithm is intentionally nebulous, since the development of this scheme is still proceeding. As mentioned in the last section, the question of when to update the modes is the subject of present research.

V. Computational Costs of Reduced Algorithm

In order to solve the equations of motion (either full or reduced), some numerical integration scheme such as Newmark's Method (see [9]) is generally employed. For the full (unreduced) set of coordinates, the stiffness matrix must be factored at each time step: this operation is of the order $n*b^2$, where n is the number of equations and b is an average bandwidth of the stiffness matrix. For a reduced coordinate system, the stiffness matrix must be factored each time the coordinates are updated, but it is assumed that this occurs infrequently, compared to the number of time steps. The reduced stiffness has to be factored at each time step: this requires on the order of p^3 operations, where p is the number of reduced coordinates. Formation of the reduced stiffness using the least efficient means (i.e., forming the unreduced stiffness and pre- and post-multiplying by the orthogonal modal matrix) requires on the order of $n*b*p$ operations. Therefore, the most inefficient reduced formulation will require on the order of $n*b*p + p^3$ operations. If p is much less than b (which is typical of large problems, but definitely not the case for the one-dimensional examples presented below), then the reduced algorithm will be competitive with direct integration. If measures are taken to improve the computational efficiency of the reduced algorithm (see [10] for one example), these asymptotic operation counts for reduced analyses should improve relative to the direct formulation.

When iteration within each time step is considered, the comparison of computational cost is less clear. At this stage of the research, it is premature to draw conclusions as to whether the reduction of coordinates enhances or retards convergence of a given iteration scheme. Development of a robust reduced iteration scheme for satisfaction of the equations of equilibrium at the end of each time step is presently one of the main topics of study by the authors. The small order of the reduced equations makes certain iteration schemes (such as those in the Newton family) more feasible.

VI. Example Problems

Several one-dimensional problems are analyzed in order to calibrate and test the reduced coordinate models. For each test problem, three types of analysis are performed:

  1) Direct integration of the equations of motion

  2) Integration of the reduced system,

      using exact eigenvectors for reduction

  3) Integration of the reduced system,

      using Lanczos vectors for reduction

In the second and third cases, the reduced coordinate algorithm presented in Section IV is used. In all cases, the equations of motion are numerically integrated using Newmark's Method (see [9]), with integration parameters:

alpha = 1/4

delta = 1/2

The first example, used for calibration purposes, is that of free vibration of a linear string. The assumption of a linear displacement field over each element leads to an initial value problem in the form of Eqn 1, with a mass matrix equal to the identity, and a tridiagonal stiffness matrix:

K = tridiag(-1, 2, -1)

The tridiagonal stiffness is equal to the so-called Jacobi matrix: the eigensystem of this matrix is known in closed form, so an exact solution is available for comparison. The number of equations chosen for this problem is fifty. The fundamental period of this system is approximately 102, and the numerical schemes were integrated from t = 0 to t = 102, using time steps of unit length.

The first ten mode shapes are shown in Figures 1 and 2. Both sets of modes were obtained from the Lanczos algorithm: the exact eigenvectors can be obtained by finding the eigensystem of the tridiagonal matrix that gives the three-term recurrence used by the Lanczos scheme. Typically, in order to calculate p eigenvectors, more than p steps of the recurrence must be taken, so finding eigenvectors using the Lanczos method is always more expensive than finding an equal number of Lanczos vectors. A vector with each component set equal to unity was used to start the algorithm. Note that antisymmetric modes appear in both cases. Since the stiffness and mass are persymmetric (symmetric about both diagonals), if a symmetric

starting vector is used to generate modes, in the absence of round-off error, only symmetric modes would be produced. Obviously, one effect of round-off error is to include antisymmetric modes that do not participate in the solution.

Because the problem is one-dimensional, the history of the displacement can be visualized in three dimensions: space, time and displacement. Figure 3 shows the solution domain in the space-time plane, and the viewpoint for display of the displacement history. One of the main reasons for choosing one-dimensional problems for test purposes (other than low computational cost) is that the entire history of the problem can be summarized in the display of displacement as a function of space and time.

Figures 4 thru 7 display displacement histories for the various methods: exact solution, direct integration, eigenvectors for reduced coordinates, and Lanczos vectors for reduced coordinates. Figures 8 and 9 show the displacement errors for the three numerical integration schemes, including results using six and ten reduced coordinates. These errors are tabulated in Table 1, along with estimates of computational costs. Note that since the problem is tridiagonal, the reduced algorithm is placed at a serious disadvantage, for the reasons given in the section on computational costs. This disadvantage should be reversed for two and three-dimensional problems.

The second problem considered is the forced vibration of a nonlinear string. In this case, the exact solution is chosen to be a product of a spatial term and a temporal factor:

$$x_i = [i/(n+1)]^*\sin(t) \qquad \text{for } i <= (n+1)/2$$

$$x_i = [(n-i)/(n+1)]^*\sin(t) \qquad \text{for } i > (n+1)/2$$

This displacement pattern is a symmetric bilinear function of the spatial coordinate, with zero displacement at the ends of the string, and a maximum value of 1/2 at the center of the string. Again, the string was subdivided into fifty point masses, and six modes were used in the reduced algorithms.

The nonlinearity present in this problem is a simple quadratic nonlinearity on the diagonal of the stiffness matrix:

$$K = \text{tridiag}(-1, 2^*(1+x_i^2), -1)$$

Given this construction of the exact solution, the accuracy of the numerical schemes can be evaluated. The analysis was performed with three different time steps, and the errors are presented in Table 2. Displacement errors associated with the shortest time steps are displayed in Figure 10. Note that the most serious errors are present at the center of the string: this is due to the "kink" in the exact solution at the this point. This disturbance propagates along the characteristics in the fully continuous problem, but is "washed-out" by a numerical procedure (such as those used in this example) that ignores the presence of characteristic curves. (This phenomenon is similar to what occurs when attempting to solve hyperbolic partial differential equations using finite-difference methods.) Away from the center of the string, the solutions are quite accurate.

A simple nonlinear correction scheme of subtracting the residual at the ith step from the load vector applied at the $(i+1)^{st}$ step was used with considerable success in this problem. Further iteration within the time step decreased the error for the direct method (refer to Table 2), but did not improve the reduced algorithms.

The last example presented is the problem of free vibration of a nonlinear string. The initial conditions are the same as in the last problem, but the nonlinearity is weaker:

$$K = \text{tridiag}(-1, 2+x_i^2, -1)$$

The results of the three integration schemes are shown in Figures 11, 12 and 13. Maximum values of the displacement, velocity, and acceleration are given in Table 3, along with more refined values obtained by direct integration using a small time step ($\Delta t = 0.01$).

VII. Conclusions and Recommendations for Further Study

This research is intended only as a first step towards a general algorithm for the solution of time-dependent nonlinear problems of the form given by Eqn 9. Considerable work needs to be done in a variety of areas in order to achieve a robust, efficient alternative to the direct integration of the full equations of motion. The example problems presented are limited in scope and geometry, and a number of two and three-dimensional problems involving nonlinear geometric

and/or material behavior are being considered. These more realistic problems, and the topics considered next, form the outline of the Ph.D. research of the first author.

There are four areas in which further research and development is presently being concentrated:

1) Development of efficient and reliable iteration schemes

The numerical integration of the nonlinear equations of motion across a time step is equivalent to the solution of a system of nonlinear equilibrium equations at the end of the time step. This solution is obtained using some sort of iteration strategy, involving formation of the equations of motion at the end of the step, and checking for a residual error. Because one of the most expensive operations involved in the proposed algorithm is the formation of the reduced stiffness at each iteration, the number of iterations required to satisfy the equilibrium equations must be kept to a minimum in order to achieve the highest computational efficiency. This requires that the iteration scheme have rapid convergence characteristics. Present research is being focused on the Newton family of methods as an alternative to a iterative application of Newmark's Method. Particular attention will be given to the use of the algorithm with comprehensive material inelasticity models, such as the bounding surface model for soils [11]. Means will be sought for combining the inherent need of such models for iteration and integration across a given time step with the iteration requirement of the dynamic analysis.

2) Increased efficiency in formation of reduced problem

The limiting factor in the present implementation of the reduced algorithm is the process of forming the reduced matrices. There are a number of avenues open for refinement of this process, including an asymptotic approach (see [10]), the construction of the reduced matrices at the element level, and the use of parallel processing. Progress on this topic, coupled with an efficient iteration scheme as mentioned in the last section, would greatly speed the proposed algorithm.

3) Adaptive determination of number of modes required

The appropriate number of modes required for an acceptable solution may not be apparent to the analyst either before or during the solution process. In addition, the determination of whether the reduced coordinates need to be updated must be automated, in order to be able to use the proposed algorithm in any way resembling the use of a 'black box'. Both these issues need to be addressed before the reduced algorithm can compete with direct integration schemes, and both require further research into the effect of projection error in the reduction process.

4) Modifications for stiff (i.e. soil-structure) systems

Preliminary results from this research indicate considerable promise for the reduced algorithm for solving certain stiff problems. In particular, soil-structure interaction problems are characterized by such a wide range of frequencies that a representative time scale

for the analysis of the coupling between the soil and structural components is difficult or impossible to find. One promising possibility would involve projecting the soil-structure problem onto the soil deformation modes that impart energy to the structure, and the lowest modes of vibration of the structure itself. The Lanczos algorithm has been used by the authors with an origin shift to obtain reduced coordinates for such problems, and work is underway on incorporating these coordinates into the proposed algorithm.

In conclusion, the use of a reduced set of modal coordinates offers an alternative to direct integration of the full equations of motion. The research presented in this work represents a first step towards an efficient implementation of this reduction for a variety of nonlinear problems.

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Appendix 1: Table of Symbols

Symbol	Definition
b	Average bandwidth of stiffness matrix
D	Diagonal matrix of squared frequencies
f	Finite element load vector
g	Reduced load vector
h	Component of f orthogonal to g
K	Finite element stiffness matrix
M	Finite element mass matrix
n	Number of equilibrium equations
p	Number of reduced coordinates
q_i	Individual trial mode shape
Q_p	Orthogonal matrix of trial mode shapes
R	Reduced stiffness matrix
t	time variable
U_p	Orthogonal matrix of exact eigenvectors
w_i	Modal natural frequency
x	Vector of nodal displacements
$\dot{x}$	Vector of nodal velocities
$\ddot{x}$	Vector of nodal accelerations
z	Vector of reduced coordinates
$\ddot{z}$	Vector of reduced accelerations

Table 1: Errors in Solution of Linear Free Vibration Problem

Type of Analysis	Maximum Absolute Errors			Cost (cpu-sec)
	Displacement	Velocity	Acceleration	
Direct integration	0.0424	0.0203	0.0139	9.53
6 Eigenvectors	0.0433	0.0173	0.0126	35.70
10 Eigenvectors	0.0426	0.0210	0.0145	80.70
6 Lanczos vectors	0.0413	0.0202	0.0137	32.63
10 Lanczos vectors	0.0465	0.0169	0.0118	77.14

Table 2: Errors in Solution of Nonlinear Forced Vibration Problem

Type of Analysis	Maximum Absolute Errors			Cost (cpu-sec)
	Displacement	Velocity	Acceleration	
Direct, 128 steps	0.0655	0.0860	0.1092	18.71
Direct, 64 steps	0.1014	0.1411	0.1894	9.85
Direct, 32 steps	0.2101	0.1749	0.1837	5.57
Direct w/ iteration,128 steps	0.0098	0.0107	0.0128	25.74
Direct w/ iteration, 64 steps	0.0253	0.0277	0.0321	13.59
Direct w/ iteration, 32 steps	0.0942	0.0964	0.1083	7.90
6 Eigenvec., 128 steps	0.0440	0.0434	0.0698	65.61
6 Eigenvec., 64 steps	0.0751	0.0673	0.1243	34.71
6 Eigenvec., 32 steps	0.2016	0.1975	0.1936	19.62
6 Lanczos vec., 128 steps	0.0371	0.0313	0.0542	63.42
6 Lanczos vec., 128 steps	0.0644	0.0657	0.0961	31.97
6 Lanczos vec., 128 steps	0.2306	0.2015	0.2049	16.82

Table 3: Approximate Solution of Nonlinear Free Vibration Problem

Type of Analysis	Maximum Absolute Response			Cost (cpu-sec)
	Displacement	Velocity	Acceleration	
Direct integration	0.8440	0.5135	0.5982	12.94
Direct w/ small step	0.8177	0.4902	0.5521	81.21
6 Eigenvectors	0.8572	0.5601	0.6674	60.41
6 Lanczos vectors	0.8650	0.5651	0.6762	57.79

(All computation times are for a VAX 11/750 with floating-point hardware)
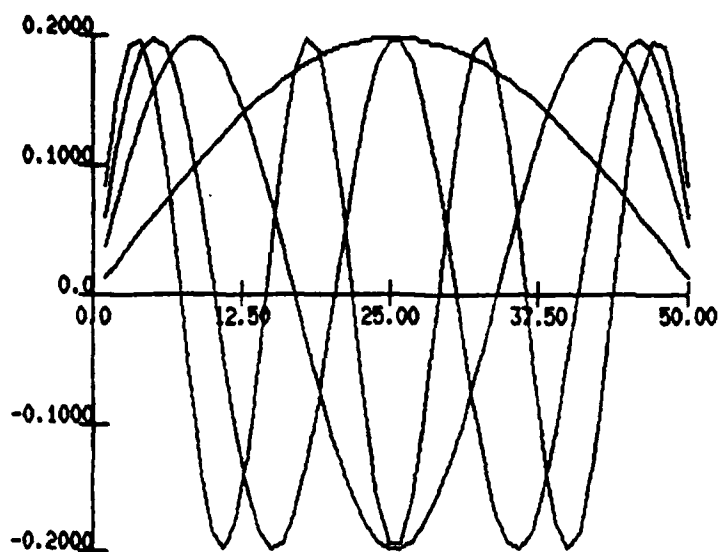
Figure 1

Exact eigenvectors
for example 1
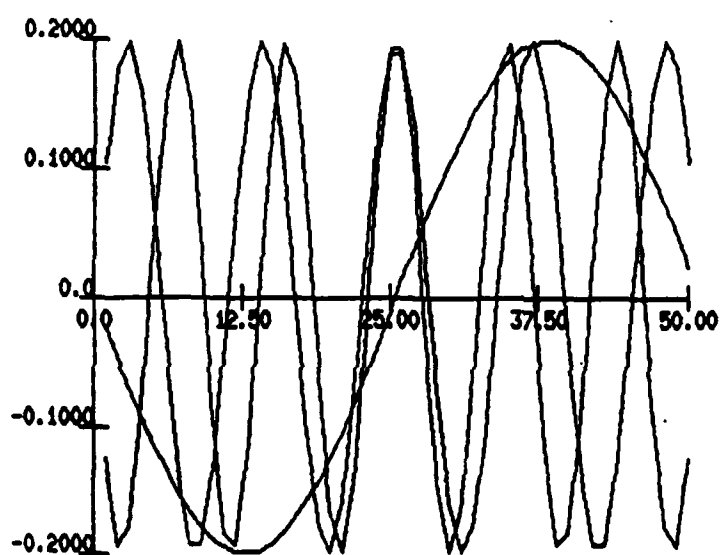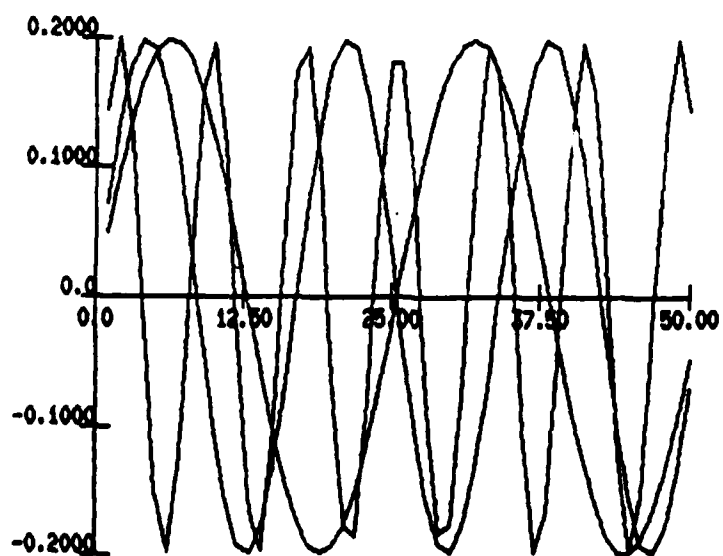
(a) Modes 1 - 4

(b) Modes 5 - 7
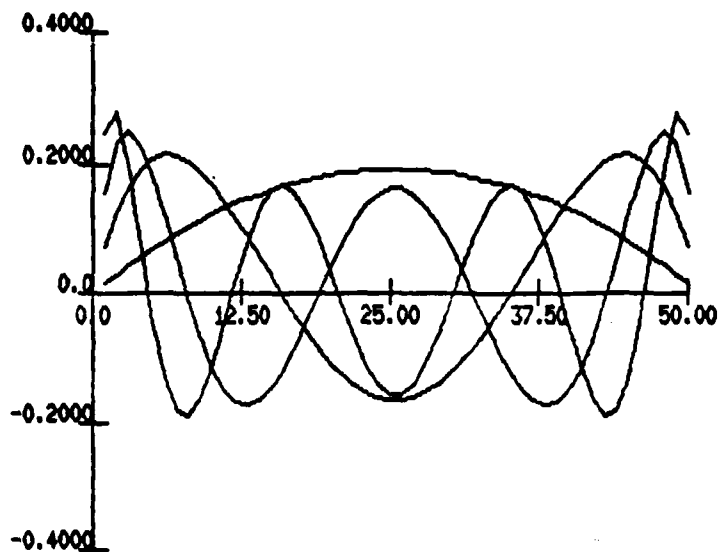
(c) Modes 8 - 10

Figure 2

Lanczos vectors
for example 1

(a) Modes 1 – 4
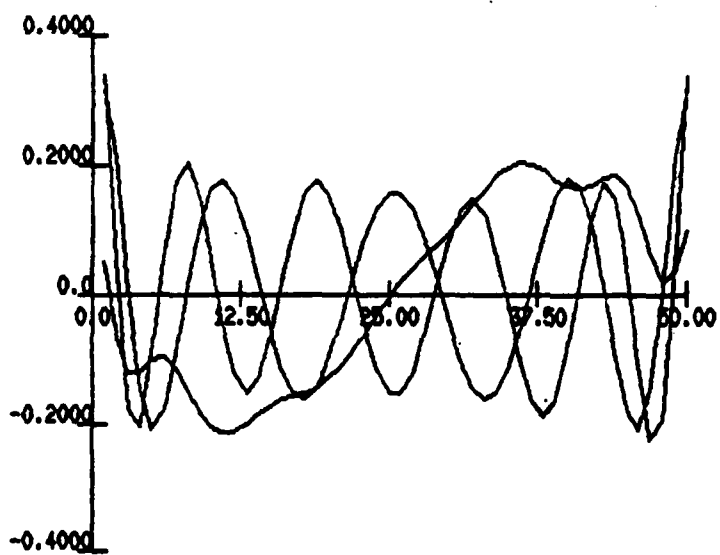
(b) Modes 5 – 7
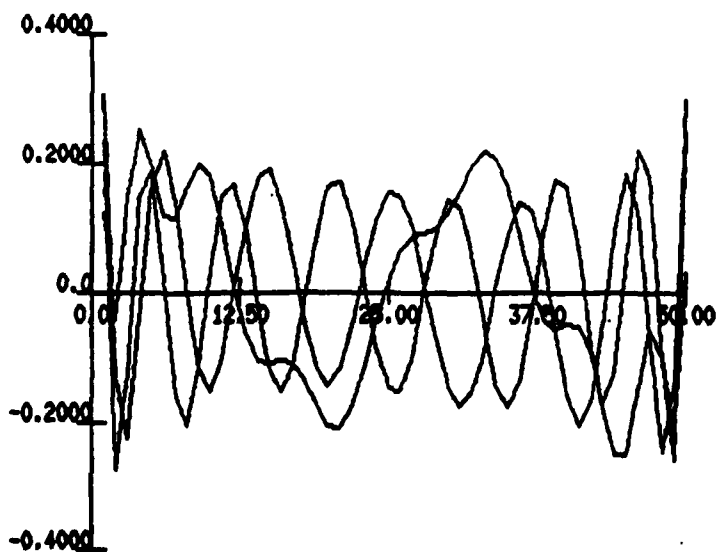
(c) Modes 8 – 10

Figure 3

One-dimensional solution history

(a) Solution domain

time

t = t_max

t = 0

space

i = 0   i = n+1

Displacement
Velocity
Acceleration
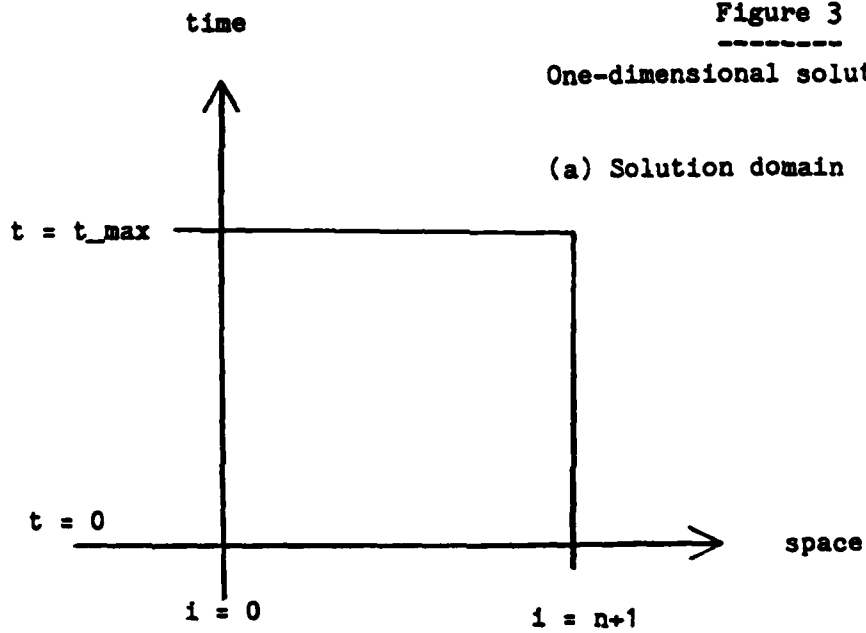
(b) Geometry of
    solution display

time

space
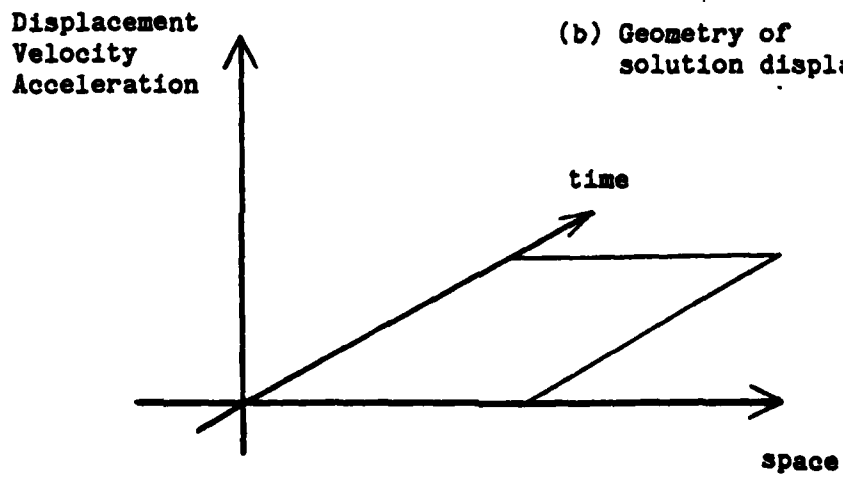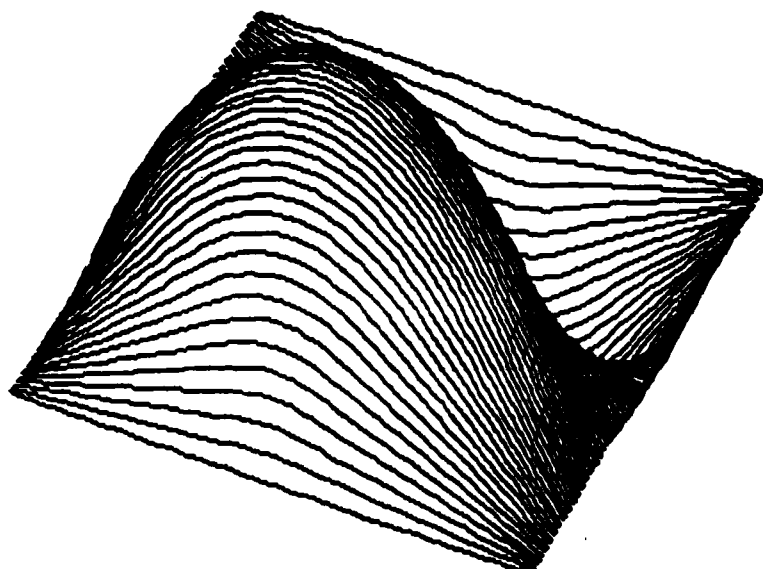
Figure 4

Exact solution history
for the linear
free-vibration problem

(a) Exact displacement
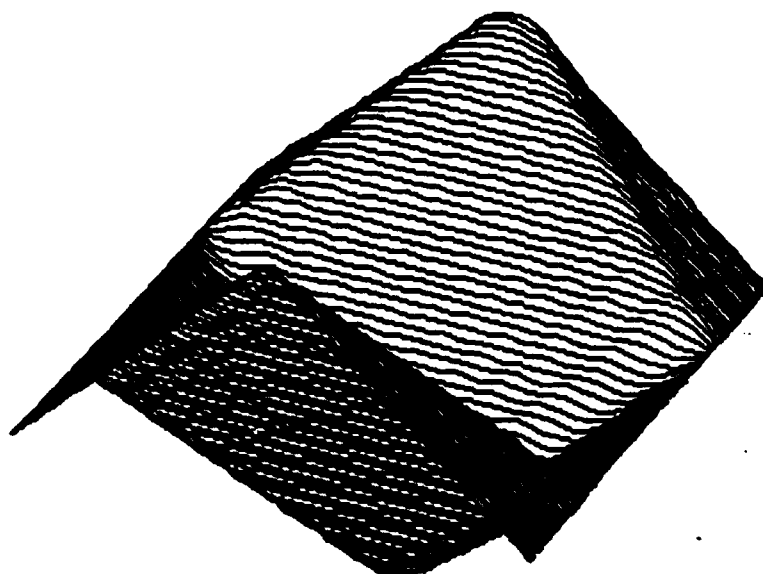
maximum absolute displacement :   6.371

(b) Exact velocity

maximum absolute velocity :   0.4902

(c) Exact acceleration
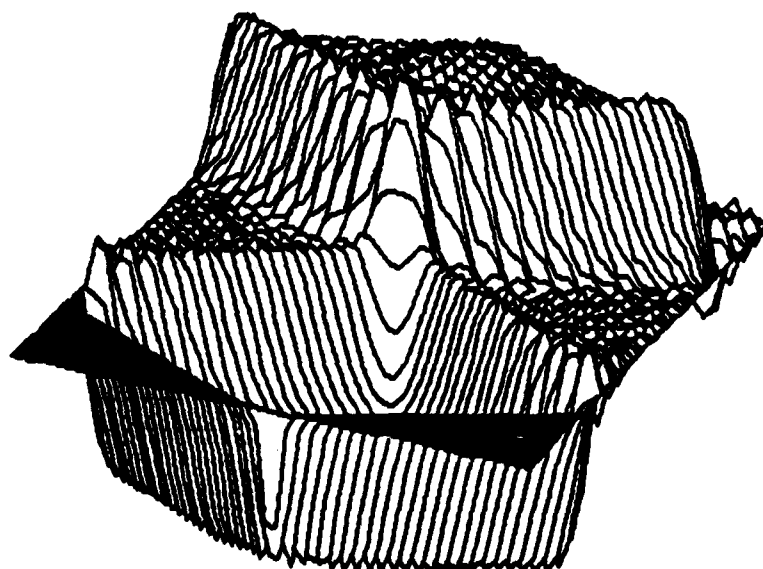
maximum absolute acceleration :   0.02823
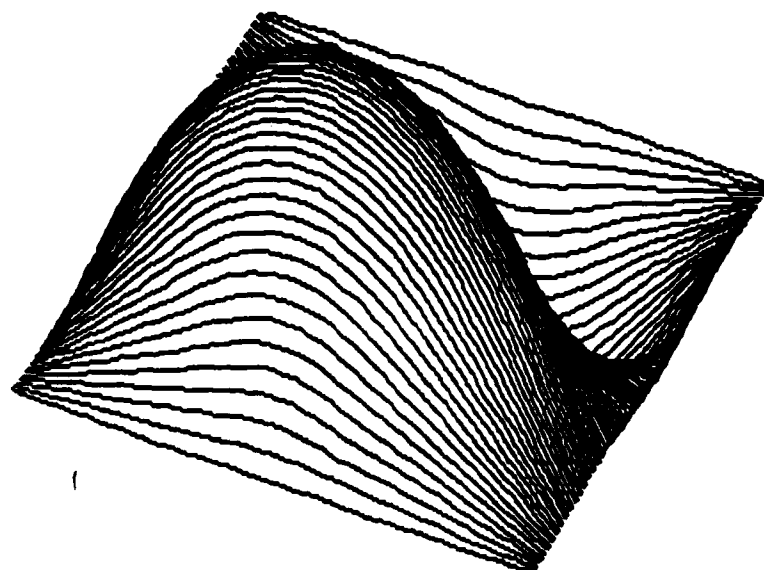
Figure 5

Numerical solution history
for the linear
free-vibration problem
(Direct integration)

(a) Approximate displacement

maximum absolute displacement : 6.370

(b) Approximate velocity
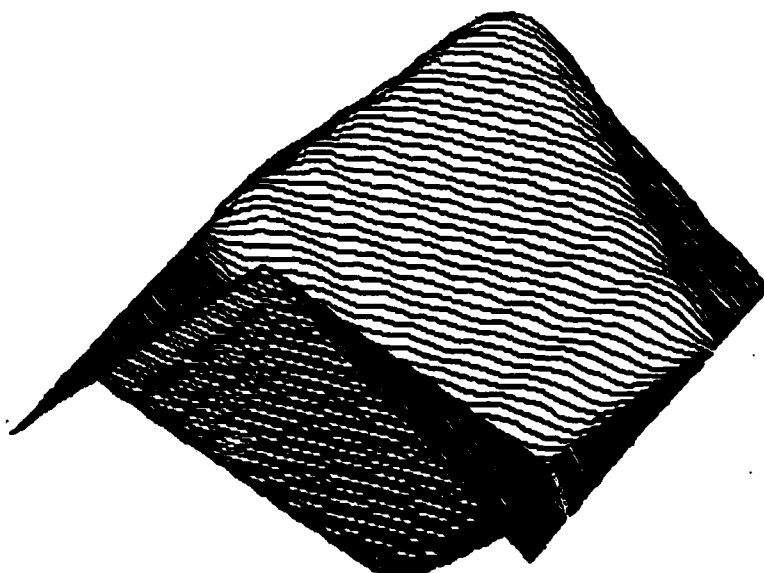
maximum absolute velocity : 0.4902

(c) Approximate acceleration
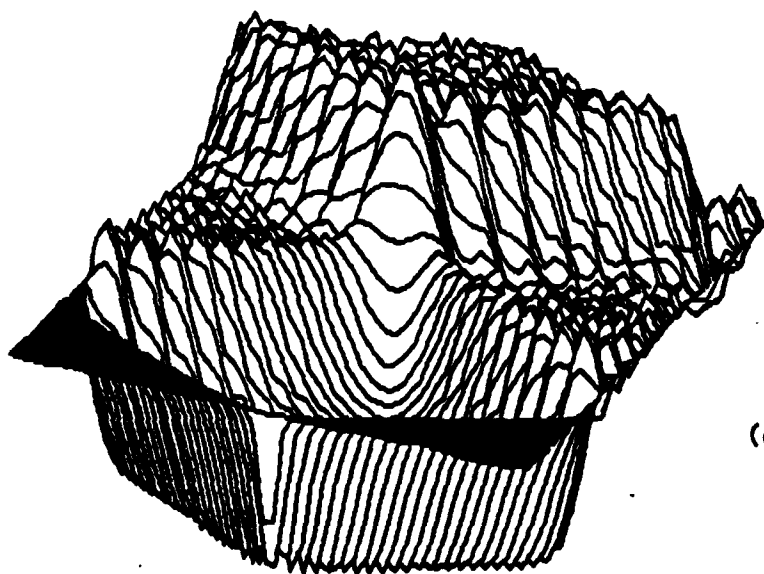
maximum absolute acceleration : 0.02980
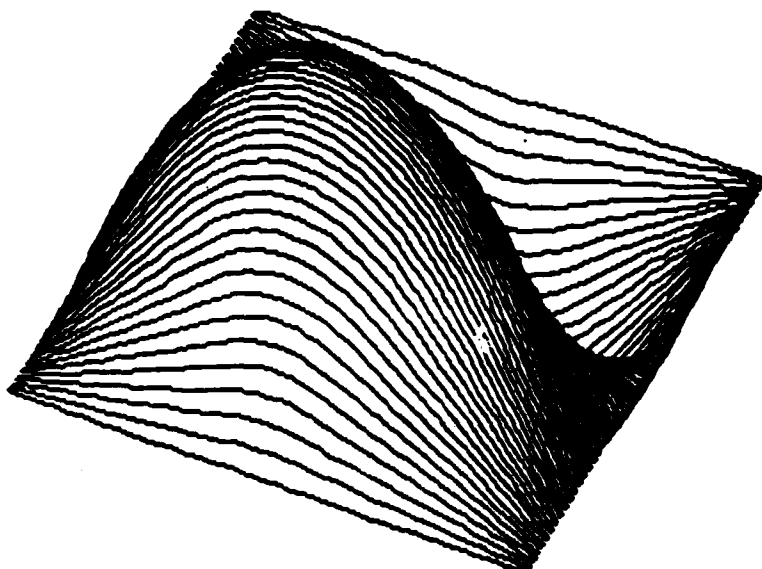
Figure 6

Numerical solution history
for the linear
free-vibration problem
(6 Eigenvectors)

(a) Approximate displacement

maximum absolute displacement : 6.372

(b) Approximate velocity
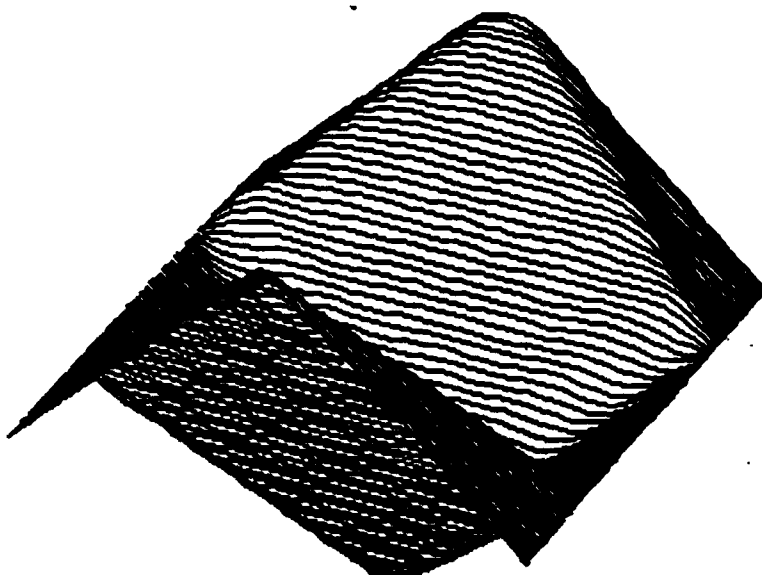
maximum absolute velocity : 0.4902

(c) Approximate acceleration
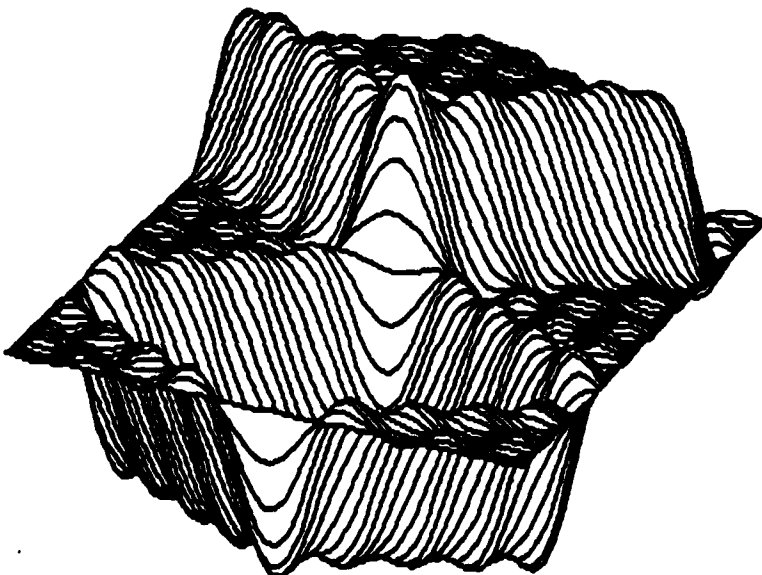
maximum absolute acceleration : 0.02606
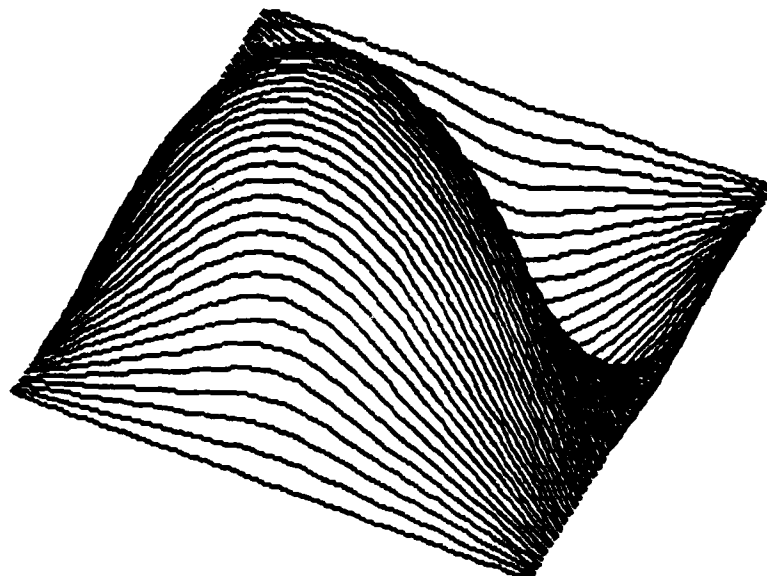
Figure 7

Numerical solution history
for the linear
free-vibration problem
(6 Lanczos vectors)

(a) Approximate displacement

maximum absolute displacement : 6.370

(b) Approximate velocity
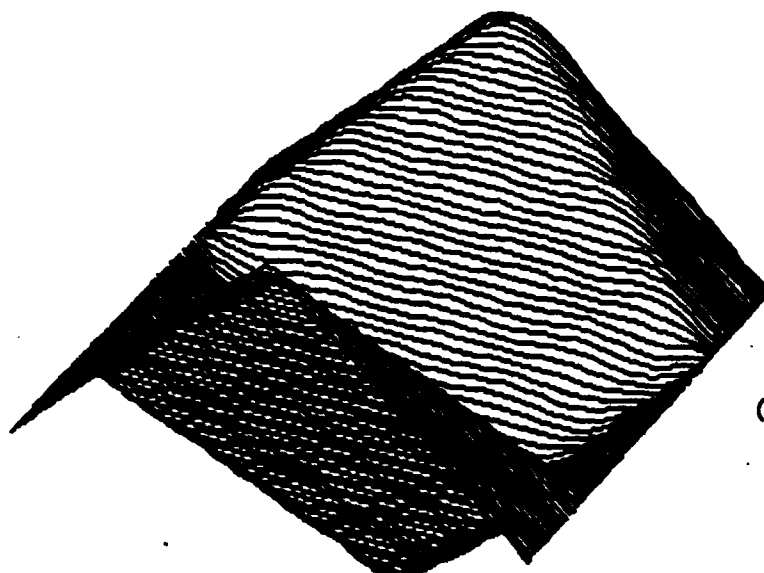
maximum absolute velocity : 0.4902

(c) Approximate acceleration
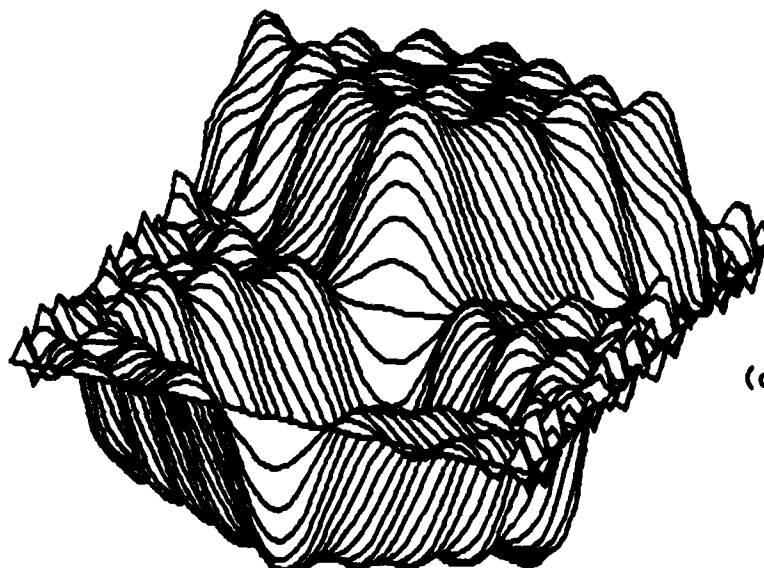
maximum absolute acceleration : 0.02473
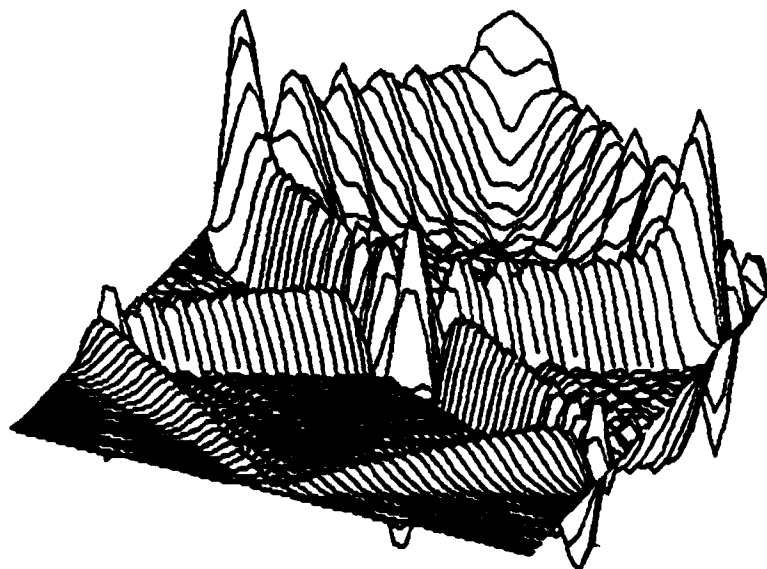
Figure 8

Displacement errors
for the linear
free-vibration problem

(a) Direct integration
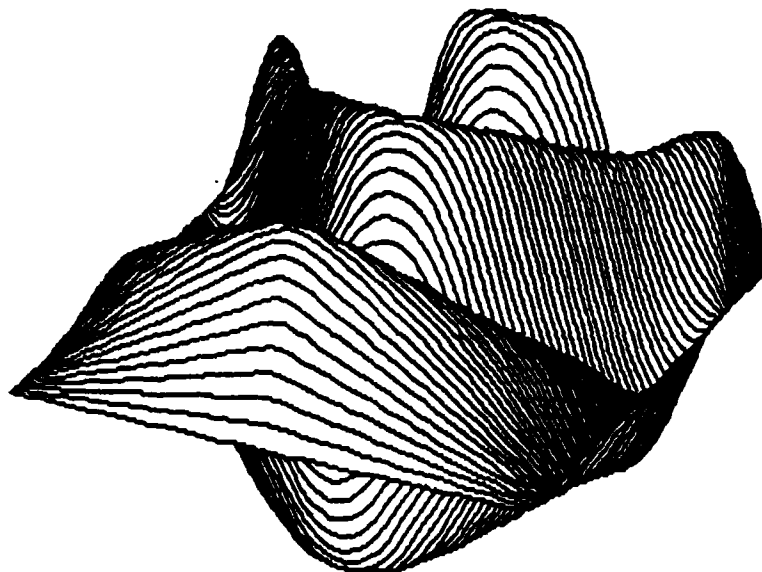
maximum absolute displacement : 0.04238
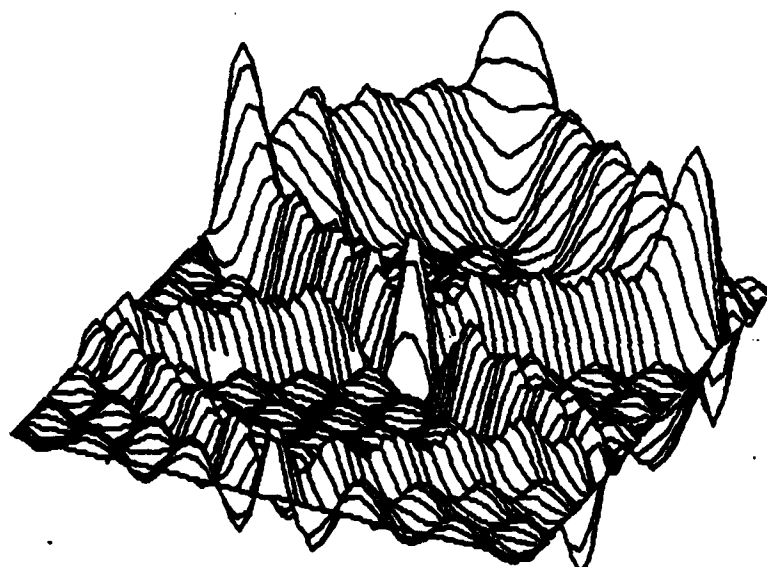
(b) 6 Eigenvectors

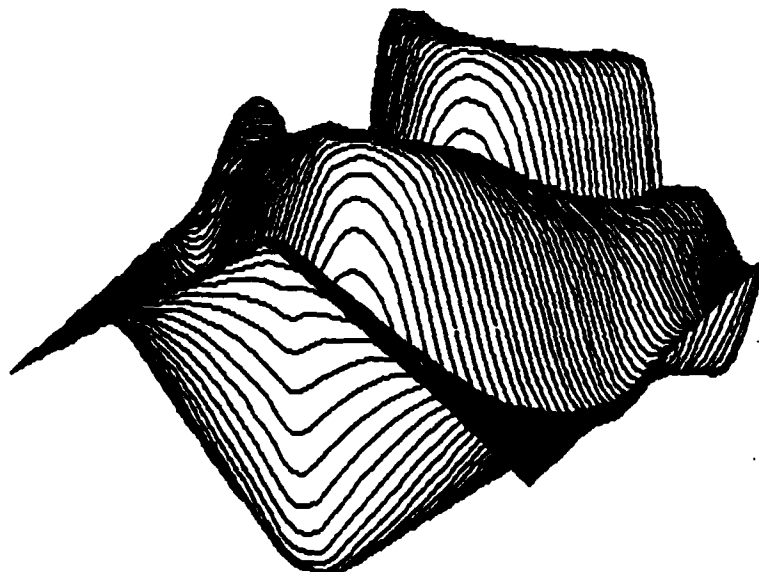maximum absolute displacement : 0.04326
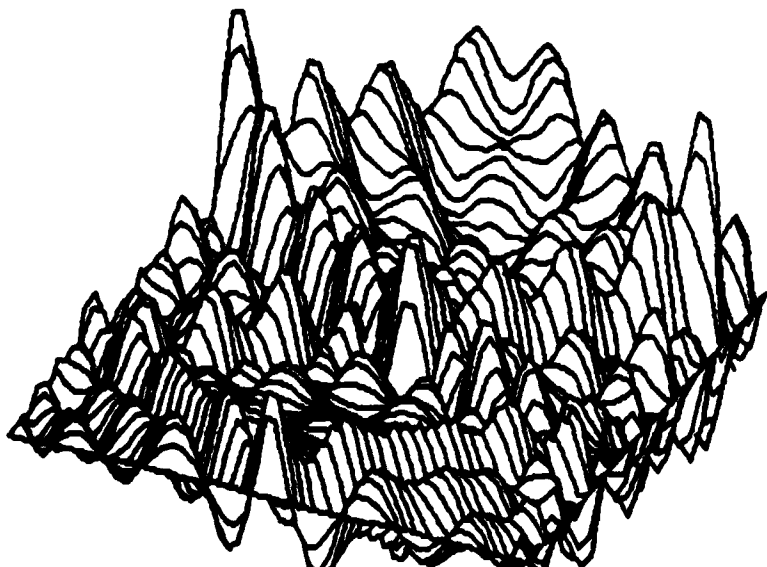
(c) 6 Lanczos Vectors

maximum absolute displacement : 0.04129

maximum absolute displacement : 0.04238

maximum absolute displacement : 0.04256
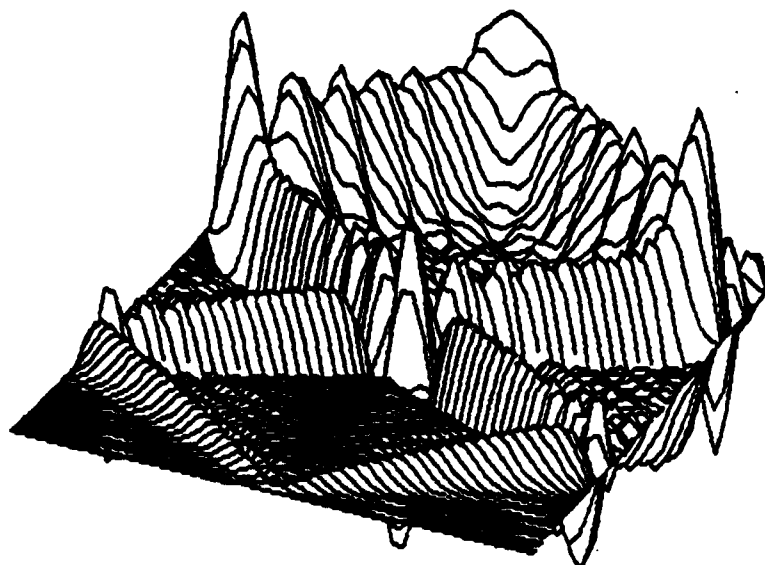
maximum absolute displacement : 0.04652

Figure 9

Displacement errors
for the linear
free-vibration problem
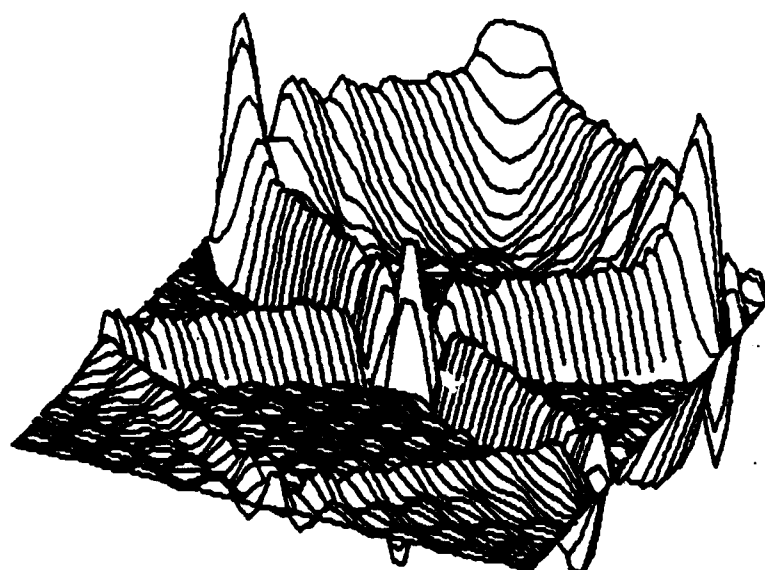
(a) Direct integration

(b) 10 Eigenvectors
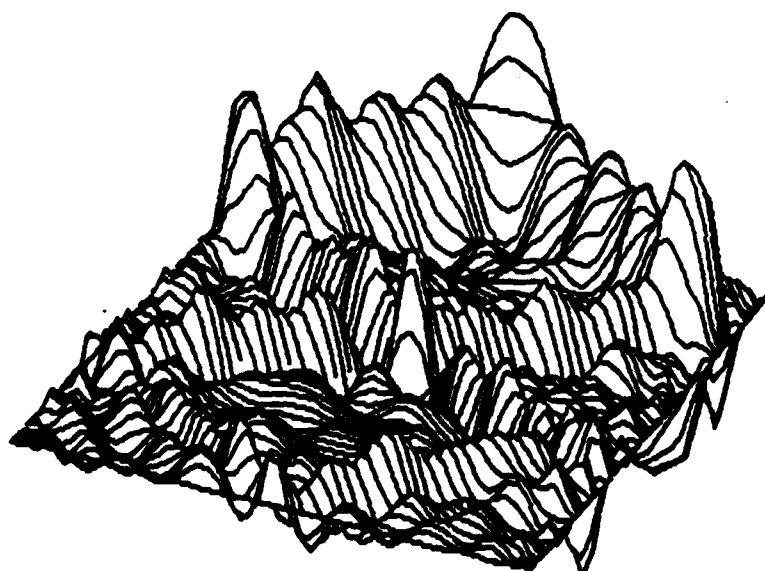
(c) 10 Lanczos Vectors
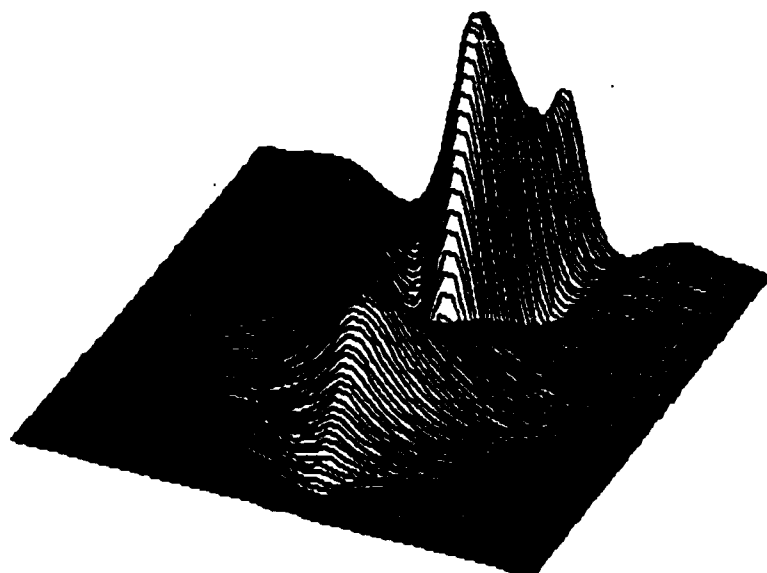
Figure 10

Displacement errors
for the nonlinear
forced vibration problem
(example 2)

(a) Direct integration

maximum absolute displacement : 0.06552
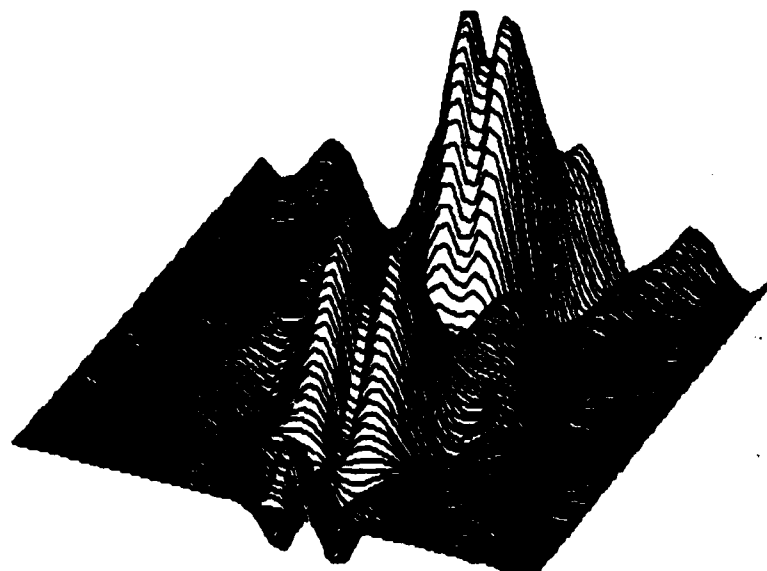
(b) 6 Eigenvectors

maximum absolute displacement : 0.04403
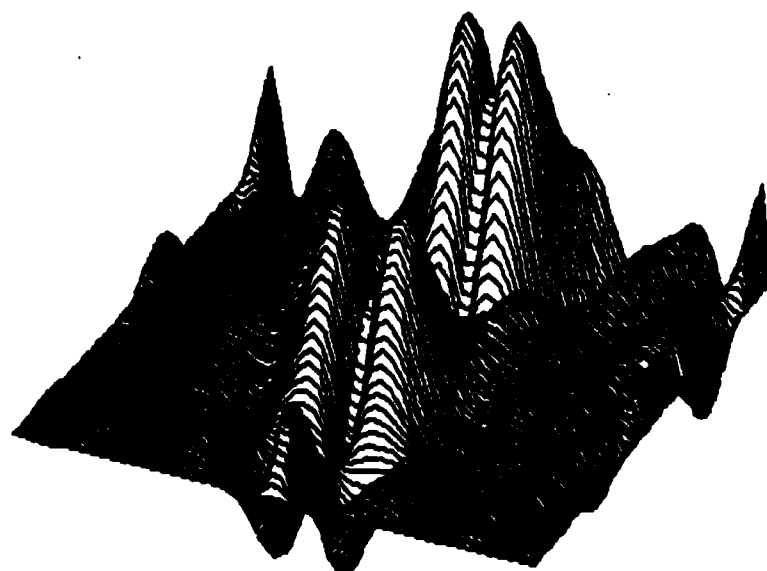
(c) 6 Lanczos Vectors
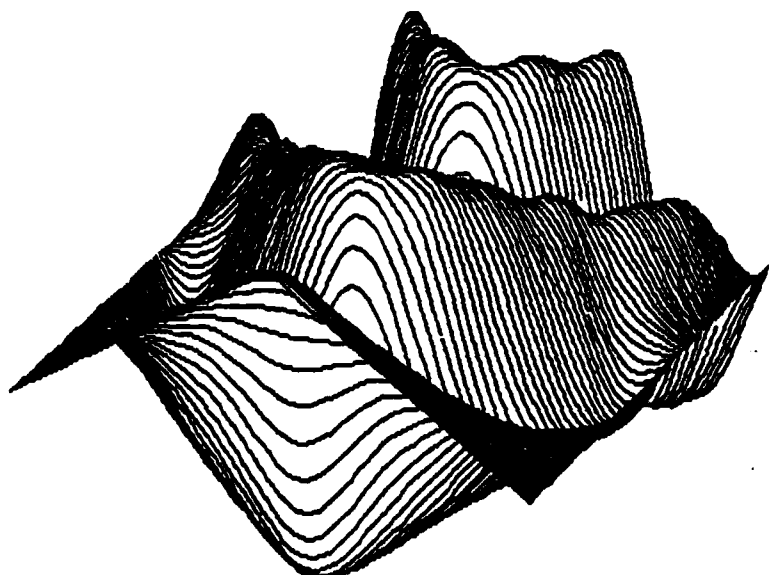
maximum absolute displacement : 0.03708

Figure 11

Numerical solution history
for the nonlinear
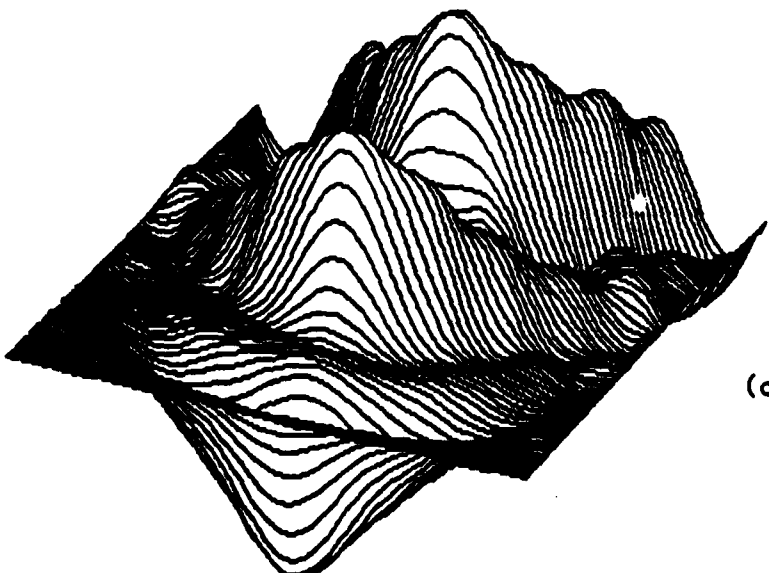free-vibration problem
(Direct integration)

(a) Approximate displacement

maximum absolute displacement : 0.8440

(b) Approximate velocity

maximum absolute velocity : 0.5135

(c) Approximate acceleration
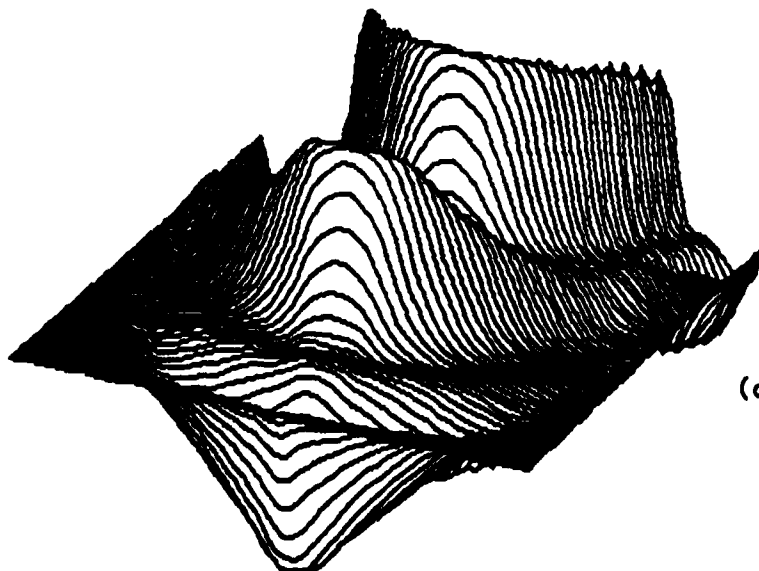
maximum absolute acceleration : 0.5862

Figure 12

Numerical solution history
for the nonlinear
free-vibration problem
(6 Eigenvectors)
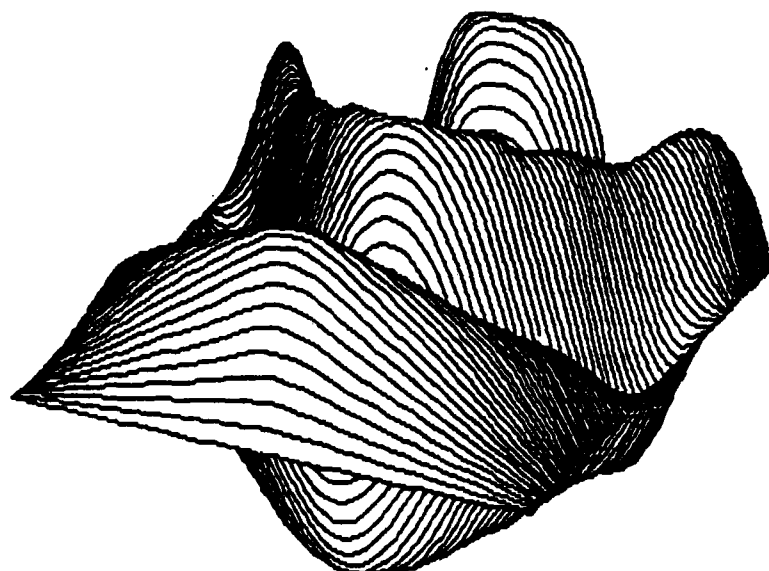
(a) Approximate displacement
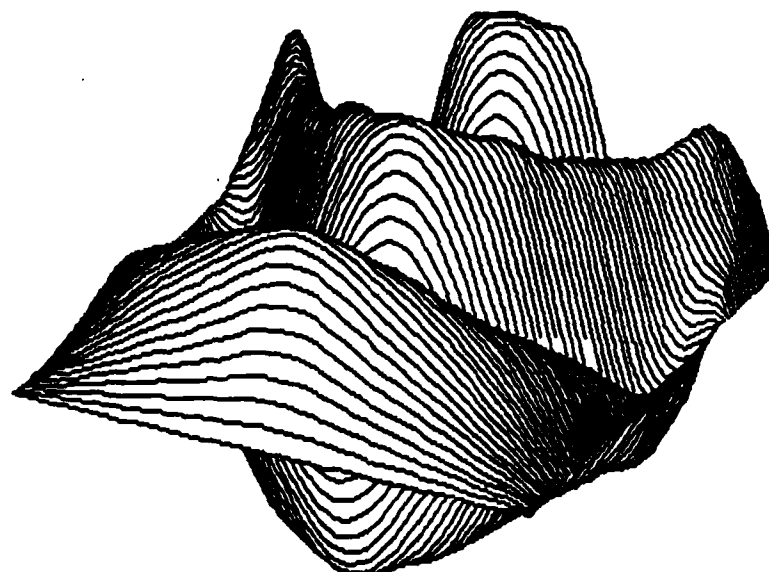
maximum absolute displacement : 0.8572

(b) Approximate velocity

maximum absolute velocity : 0.5601
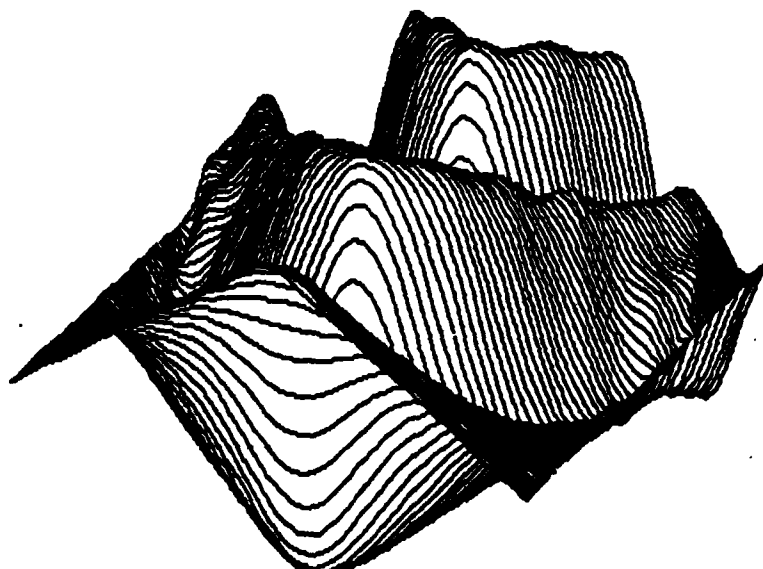
(c) Approximate acceleration
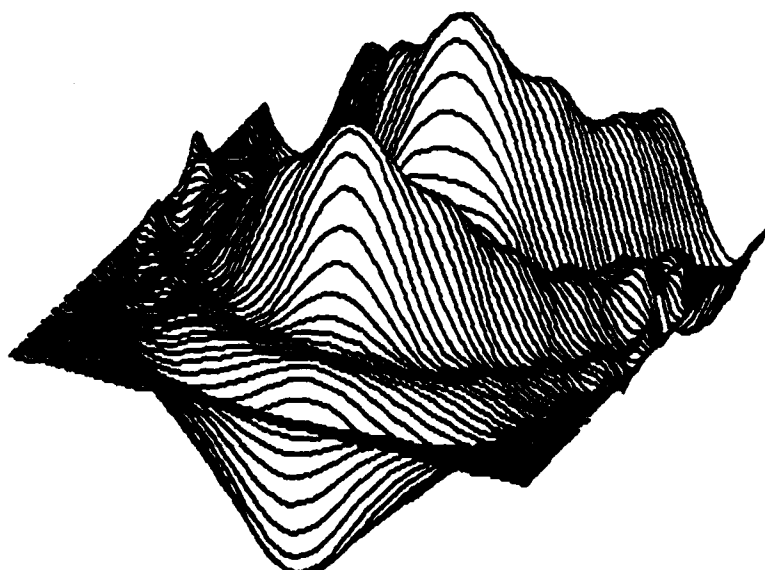
maximum absolute acceleration : 0.6874

Figure 13
—————————
Numerical solution history
for the nonlinear
free-vibration problem
(6 Lanczos vectors)

(a) Approximate displacement

maximum absolute displacement : 0.8650

(b) Approximate velocity

maximum absolute velocity : 0.5651

(c) Approximate acceleration

maximum absolute acceleration : 0.6762

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